EQUILATERAL TRIANGLES IN $\mathbb{Z}^4$

EUGEN J. IONASCU

Abstract. We give a characterization of all three points in $\mathbb{R}^4$ with integer coordinates which are at the same Euclidean distance apart. In three dimension the problem is characterized in terms of solutions of the Diophantine equations $a^2+b^2+c^2=3d^2$. In $\mathbb{R}^4$, the characterization is essentially based on two different solutions of the same equation.

1. INTRODUCTION

It is known (see [2], [7]) that the Diophantine equation

$$a^2+b^2+c^2=3d^2 \tag{1}$$

has always non-trivial solutions ($\gcd(a,b,c)=1$ and $0<a\le b\le c$, these will be called *primitive* solutions) for every odd number d. As a result of this investigation, we will also see a simple argument for this statement. This equation is at the heart of our main result so let us remind the reader a few facts about it. In general the equations (1) has solutions for which $\gcd(a,b,c)>1$ in addition to the primitive ones, for instance

$$3(15^2)=5^2+5^2+25^2=3^2+15^2+21^2=1^2+7^2+25^2=5^2+11^2+23^2=5^2+17^2+19^2,$$

and we will see that all these solutions are important in our considerations. If we are interested in integer solutions of (1) then, of course, one can permute $[a,b,c]$ in six different ways and change the signs, to obtain a total of 48, in general (if a, b and c are all different), distinct ordered triples. If d is odd, it is easy to see that a, b and c must be all odd numbers too. One way to obtain solutions of (1) is to look at the factorization of the number $3d^2-c^2$ and check if it satisfies the characterization of being a sum of two perfect squares. As an example, if $d=7$, $3(49)-1=146=2(73)$ and the prime $73=4(18)+1$ implies we can write $3(49)=1^2+5^2+11^2$. The number of primitive solutions is given by

$$\pi\epsilon(d):=\frac{\Lambda(d)+24\Gamma_2(d)}{48}, \tag{2}$$

where

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$$\Gamma_2(d) = \begin{cases} 0 \ \ \text{if d is divisible by a prime factor of the form 8s+5 or 8s+7}, \ \ s \geq 0, \\ \\ 1 \ \ \text{if d is 3} \\ \\ 2^k \ \ \begin{cases} \text{where k is the number of distinct prime factors of d} \\ \\ \text{of the form 8s+1, or 8s+3 } (s > 0), \end{cases} \end{cases} \tag{3}$$

$$\Lambda(d) := 8d \prod_{p|d, p \ prime} \left(1 - \frac{(\frac{-3}{p})}{p} \right), \tag{4}$$

and $(\frac{-3}{p})$ is the Legendre symbol. We remind the reader that, if p is an odd prime then

$$(\frac{-3}{p}) = \begin{cases} 0 \ \ \text{if } p = 3 \\ \\ 1 \ \ \text{if } p \equiv 1 \text{ or } 7 \pmod{12} \\ \\ -1 \ \ \text{if } p \equiv 5 \text{ or } 11 \pmod{12} \end{cases} . \tag{5}$$

In [6] we have characterized the primitive triples $(a, c, d) \in \mathbb{N}^3$ satisfying $2a^2 + c^2 = 3d^2$. This was done in a manner similar to the way that Pythagorean triples are usually described with a one-to-one correspondence to a special set of pairs of natural numbers.

Theorem 1.1. *Suppose that k and ℓ are positive integers with k odd and $\gcd(k, \ell) = 1$. Then a, c and d given by*

$$d = 2\ell^2 + k^2 \ with \ \begin{cases} a = |2\ell^2 + 2k\ell - k^2|, \ \ c = |k^2 + 4k\ell - 2\ell^2|, \ \ if \ \ k \not\equiv \ell \pmod{3} \\ \\ a = |2\ell^2 - 2k\ell - k^2|, \ \ c = |k^2 - 4k\ell - 2\ell^2|, \ \ if \ \ k \not\equiv -\ell \pmod{3} \end{cases} \tag{6}$$

constitute a positive primitive solution of $2a^2 + c^2 = 3d^2$. Conversely, with the exception of the trivial solution $a = c = d = 1$, every positive primitive solution for $2a^2 + c^2 = 3d^2$ appears in the way described above for some l and k.

This theorem provides a very simple way of generating lots of solutions of the equation (1).

In this paper we are interested in finding a simple way of constructing three points in $\mathbb{Z}^4$, say $A = (a_1, a_2, a_3, a_4)$, $B = (b_1, b_2, b_3, b_4)$, and $C(c_1, c_2, c_3, c_4)$ with $a_i, b_i, c_i \in \mathbb{Z}$, $i = 1..4$, such that

$$\sum_{i=1}^{4} (a_i - c_i)^2 = \sum_{i=1}^{4} (b_i - c_i)^2 = \sum_{i=1}^{4} (a_i - b_i)^2 = D > 0. \tag{7}$$

We will disregard an integer translation, so we will assume that C is the origin ($C = O$). Another trivial way to get new equilateral triangles from old ones is to permute the coordinates and change signs (a total of possibly 384 different triangles). Also, we will mainly be talking about *irreducible* triangles, in the sense that the triangle OAB cannot be shrunk by an integer factor to a triangle in $\mathbb{Z}^4$.

The first example of a whole family of equilateral triangles in $\mathbb{Z}^4$ is given by any solution of (1), taking $A(d, a, b, c)$ and $B(2d, 0, 0, 0)$. The triangle OAB is irreducible as long as we start with a primitive solution of (1) and its side-lengths are equal to $2d$. This is totally different of the 3-dimensional situation. We are including here the main classification result in $\mathbb{Z}^3$ from [5].

Theorem 1.2. *The sub-lattice of all points in the plane $\mathcal{P}_{a,b,c} := \{(\alpha, \beta, \gamma) \in \mathbb{Z} | a\alpha + b\beta + c\gamma = 0\}$ contains two vectors $\overrightarrow{\zeta}$ and $\overrightarrow{\eta}$ such that the triangle $\mathcal{T}_{a,b,c}^{m,n} := \triangle OPQ$ with P, Q in $\mathcal{P}_{a,b,c}$, is equilateral if and only if for some integers m, n*

$$\overrightarrow{OP} = m\overrightarrow{\zeta} - n\overrightarrow{\eta}, \quad \overrightarrow{OQ} = n\overrightarrow{\zeta} + (m-n)\overrightarrow{\eta}, \text{ with} \tag{8}$$
$$\overrightarrow{\zeta} = (\zeta_1, \zeta_1, \zeta_2), \overrightarrow{\varsigma} = (\varsigma_1, \varsigma_2, \varsigma_3), \overrightarrow{\eta} = \frac{\overrightarrow{\zeta} + \overrightarrow{\varsigma}}{2},$$

$$\begin{cases} \zeta_1 = -\dfrac{rac + dbs}{q} \\ \\ \zeta_2 = \dfrac{das - bcr}{q} \\ \\ \zeta_3 = r \end{cases}, \quad \begin{cases} \varsigma_1 = \dfrac{3dbr - acs}{q} \\ \\ \varsigma_2 = -\dfrac{3dar + bcs}{q} \\ \\ \varsigma_3 = s \end{cases}, \tag{9}$$

where $q = a^2 + b^2$ and r, s can be chosen so that all six numbers in (9) are integers. The sides-lengths of $\triangle OPQ$ are equal to $d\sqrt{2(m^2 - mn + n^2)}$. Moreover, r and s can be constructed in such a way that the following properties are also verified:
(i) r and s satisfy $2q = s^2 + 3r^2$ and similarly $2(b^2 + c^2) = \varsigma_1^2 + 3\zeta_1^2$ and $2(a^2 + c^2) = \varsigma_2^2 + 3\zeta_2^2$
(ii) $r = r'\omega\chi$, $s = s'\omega\chi$ where $\omega = gcd(a, b)$, $gcd(r', s') = 1$ and χ is the product of the prime factors of the form $6k - 1$ of $(a^2 + b^2)/\omega^2$
(iii) $|\overrightarrow{\zeta}| = d\sqrt{2}$, $|\overrightarrow{\varsigma}| = d\sqrt{6}$, and $\overrightarrow{\zeta} \cdot \overrightarrow{\varsigma} = 0$
(iv) $s + i\sqrt{3}r = gcd(A - i\sqrt{3}B, 2q)$, in the ring $\mathbb{Z}[i\sqrt{3}]$, where $A = ac$ and $B = bd$.

When it comes to the irreducible elements in $\mathbb{Z}[i\sqrt{3}]$ we use only the following decomposition of 4 as $(1 + \sqrt{3}i)(1 - \sqrt{3}i)$. As we can see from the above theorem, the possible values of D (defined in (7)) for the existence of an irreducible triangle must be of the form $2(m^2 - mn + n^2)$, $m, n \in \mathbb{Z}$.

We will see that in $\mathbb{Z}^4$ every even D can be achieved for some equilateral triangle not necessarily irreducible.

Another family of examples is given by

$$A(a+b, a-b, c+d, c-d) \text{ and } B(a+c, d-b, c-a, -b-d), \tag{10}$$

where a, b, c and d are arbitrary integers such that $\gcd(a,b,c,d) = 1$ and not all odd. The side-lengths are $\sqrt{2(a^2+b^2+c^2+d^2)}$.

In $\mathbb{Z}^3$ ([6]) only the equilateral triangles given as in Theorem 1.2 with $m^2-mn+n^2=k^2$, $k \in \mathbb{Z}$, can be extended to a regular tetrahedron in $\mathbb{Z}^3$. The family in (10) can be completed to a regular tetrahedron by taking $C(b+c, a+d, d-a, c-b)$. We will also show that every triangle in the first family given here can be extended to regular tetrahedrons. In fact, we will show this for the more general family defined by

$$\begin{aligned} &A((m-2n)d, ma, mb, mc) \text{ and} \\ &B((2m-n)d, na, nb, nc), \text{ with } \ m,n \in \mathbb{Z},\ gcd(m.n) = 1. \end{aligned} \tag{11}$$

2. Some results

Our classification is driven essentially by the side-length $\sqrt{D}$. From (7) we see that

$$D = |\overrightarrow{AB}|^2 = \sum_{i=1}^{4} a_i^2 - 2 < \overrightarrow{OA}, \overrightarrow{OB} > + \sum_{i=1}^{4} b_i^2 \quad \Leftrightarrow \quad < \overrightarrow{OA}, \overrightarrow{OB} >= D/2.$$

So, $\sum_{i=1}^4 a_i b_i = D/2$ which proves that D must be even, and so $D = 2k$ with $k \in \mathbb{N}$. Let us assume that k is a multiple of 2. Then $D = 2k$ is a multiple of 4. Then

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 = b_1^2 + b_2^2 + b_3^2 + b_4^2 \equiv 0 \pmod 4$$

implies that either all a_i's are even or all are odd, since every perfect square is equal to 0 or 1 (mod 4). Similarly the b_i's are all even or all odd. If the triangle OAB is irreducible we may assume that a_i's are all odd numbers. In this case

$$2k = a_1^2 + a_2^2 + a_3^2 + a_4^2 = \sum_{i=1}^{4} (2c_i + 1)^2 = 4[1 + \sum_{i=1}^{4} c_i(c_i + 1)].$$

Hence $\frac{k}{2}$ must be odd. So, the power of 2 in the decomposition of D must be 1 or 2 in order to have an irreducible equilateral triangle in $\mathbb{Z}^4$ whose side-lengths are $\sqrt{D}$. We will prove the converse latter. We include next some examples for small values of k:

k	A	B	D
1	(0,0,1,1)	(0,1,0,1)	2
2	(1,1,1,1)	(1,1,1,-1)	4
3	(0,1,1,2)	(1,1,2,0)	6
3	(0,1,1,2)	(0,2,-1,1)	6
4	2(1,1,0,0)	2(1,0,1,0)	8
5	(1,1,2,2)	(3,0,1,0)	10
6	(1,1,1,3)	(-1,1,3,1)	12
7	(0,1,2,3)	(3,1,0,2)	14
7	(0,1,2,3)	(0,3,-1,2)	14
8	2(1,1,1,1)	2(1,1,1,-1)	16
9	(1,2,2,3)	(1,3,-2,2)	18
9	(0,1,1,4)	(0,1,4,1)	18
10	(1,1,3,3)	(0,4,2,0)	20

k	A	B	D
11	(1,1,2,4)	(-1,4,2,1)	22
12	2(0,1,1,2)	2(1,1,2,0)	24
12	2(0,1,1,2)	2(0,2,-1,1)	24
13	(0,1,3,4)	(4,1,0,3)	26
13	(2,2,3,3)	(2,-3,3,2)	26
14	(2,2,2,4)	(-3,-3,1,3)	28
15	(1,2,3,4)	(3,1,-2,4)	30
16	4(1,1,0,0)	4(1,0,1,0)	32
17	(1,2,2,5)	(5,-1,2,2)	34
18	(1,1,3,5)	(1,5,-1,3)	36
19	(0,2,3,5)	(5,2,0,3)	38
19	(2,3,3,4)	(2,-3,4,3)	38
20	2(1,1,2,2)	2(3,0,1,0)	40

This table suggest that we can introduce a sequence, $ET_4(k)$, which counts the number of equilateral triangles in $\mathbb{Z}^4$ which have side-lengths equal to $\sqrt{2k}$, disregarding the permutations and the change of signs. The next observation we make is a simple way to generate new triangles from given ones.

Proposition 2.1. *(a) Given an irreducible equilateral triangle OPQ in $\mathbb{Z}^4$ and m, n two integers then the triangle $OP'Q' \in \mathbb{Z}^4$ defined by*

$$\overrightarrow{OP'} = m\overrightarrow{OP} - n\overrightarrow{OQ}, \quad \overrightarrow{OQ'} = (m-n)\overrightarrow{OQ} + n\overrightarrow{OP}, \tag{12}$$

is also equilateral and the side-lengths are equal to $|PQ|\sqrt{m^2-mn+n^2}$. Moreover, $OP'Q'$ is irreducible if and only if $\gcd(m,n)=1$.

(b) There exists an equilateral triangle $ORS \in \mathbb{Z}^4$ in the same "plane" as OPQ such that all equilateral triangles in this plane are generated as in (12) from ORS.

Proof. (a) We just calculate

$$|\overrightarrow{OP'}|^2 = m^2OP^2 - 2mn < \overrightarrow{OP}, \overrightarrow{OQ} > +n^2OQ^2 = |PQ|^2(m^2-mn+n^2),$$

$$|\overrightarrow{OQ'}|^2 = (m-n)^2OP^2 + 2(m-n)n < \overrightarrow{OP}, \overrightarrow{OQ} > +n^2OQ^2 = |PQ|^2(m^2-mn+n^2) \text{ and}$$

$$|\overrightarrow{P'Q'}|^2 = |(m-n)\overrightarrow{OP} - m\overrightarrow{OQ}|^2 = |PQ|^2(m^2-mn+n^2).$$

The last statement is more or less obvious and we let it to the reader.

(b) To show the existence of ORS we choose an equilateral triangle in the same plane with the smallest side-lengths. If this triangle doesn't cover all the triangles that are in the same plane there exists one, say OAB as in Figure 2. The position of A and B within the tessellation generated by ORS provide existence of an equilateral triangle with smaller side-lengths. ∎

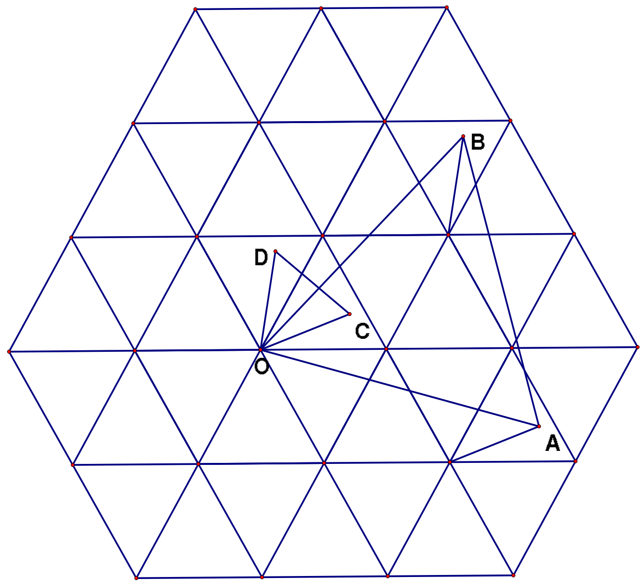

Figure 1: Two distinct tessellations

We are going to refer to the triangle given by (b) as a minimal triangle which generates OPQ.
Remark: Let us observe that we can can be sure that OPQ is itself minimal if its square of the side-lengths is not a multiple of 4 and does not have any divisors of form $m^2 - mn + n^2$ or equivalently, no prime divisors of the form 3 or $6k + 1$.

Proposition 2.2. *An irreducible equilateral triangle of side-lengths $\sqrt{D}$ exists in $\mathbb{Z}^4$ if and only if $D/2$ or $D/4$ is an odd number.*

Proof. We need to check the sufficiency. Let $k = D/2$ and consider its prime factorization. We split $k = k_1 k_2$ where k_1 is the product of all primes of the form $-1 \pmod 6$ or 2 and k_2 is the product of all other primes. By Lagrange theorem we can write $k_1 = a^2+b^2+c^2+d^2$, a, b, c and d in $\mathbb{Z}$. Using (10) we obtain such an equilateral triangle which by the remark following Proposition 2.1 must be irreducible. We take one representation of $k_2 = m^2 - mn + n^2$ such that $\gcd(m, n) = 1$ and apply Proposition 2.1 to the triangle constructed above. We obtain an equilateral which is irreducible of side-lengths $\sqrt{k_1 k_2} = \sqrt{D}$. ■

3. A necessary condition

Using the Lagrange's Identity,

$$D^2 = 4k^2 = (\sum_{i=1}^{4} a_i^2)(\sum_{i=1}^{4} b_i^2) = (\sum_{i=1}^{4} a_i b_i)^2 + \sum_{1\le i<j\le 4} (a_i b_j - a_j b_i)^2.$$

This implies that

$$\sum_{1\le i<j\le 4}(a_ib_j-a_jb_i)^2=3k^2. \tag{13}$$

If we denote by $\Delta_{ij}=(-1)^{i-j}(a_ib_j-a_jb_i)$, we observe that $\Delta_{ij}=\Delta_{ji}$ for all $i,j\in\{1,2,3,4\}$, $i<j$. We have a relation which is a constrain on the solutions of (13):

$$\Delta_{12}\Delta_{34}-\Delta_{13}\Delta_{24}+\Delta_{14}\Delta_{23}=0. \tag{14}$$

This implies that we also have a writing similar to the one in (13) but with only three perfect squares:

$$3k^2=(\Delta_{12}\pm\Delta_{34})^2+(\Delta_{13}\mp\Delta_{24})^2+(\Delta_{14}\pm\Delta_{23})^2. \tag{15}$$

From the three dimensional situation we know that this equation has solutions for every k odd. The analogy carries over even further if we observe that

$$\begin{cases} a_1(0)+a_2\Delta_{34}+a_3\Delta_{24}+a_4\Delta_{23}=0\\ b_1(0)+b_2\Delta_{34}+b_3\Delta_{24}+b_4\Delta_{23}=0\\ a_1\Delta_{23}+a_2\Delta_{13}+a_3\Delta_{12}+a_4(0)=0\\ b_1\Delta_{23}+b_2\Delta_{13}+b_3\Delta_{12}+a_4(0)=0\end{cases} \tag{16}$$

One may couple our problem with the existence of orthogonal matrices having rational coordinates. The group of symmetries of the four-dimensional space which leaves the lattice $\mathbb{Z}^4$ invariant is not something that we are interested in since that is just permuting the variables or changing their signs. The first interesting orthogonal matrix we may look at is

$$O_2=\frac{1}{4}\begin{pmatrix}1&1&1&1\\1&1&-1&-1\\1&-1&1&-1\\1&-1&-1&1\end{pmatrix},$$

and the quaternion theory provide a huge class of such examples. One can take such a matrix and use it to transform equilateral triangles in $\mathbb{Z}^3$ into equilateral triangles in $\mathbb{Z}^4$ but that changes their sides by a factor n^2.

Let us observe that in the example (10) we have $\Delta_{12}=a^2+ac-bc-ad-db+b^2$, $\Delta_{34}=c^2-ac+ad+bc+db+d^2$, $\Delta_{13}=-a^2+bc-ab-c^2-ad-cd$, $\Delta_{14}=ac+c^2-cd+ab+b^2+db$, $\Delta_{23}=cd+d^2-db-ac+a^2-ab$, and $\Delta_{24}=-ab-ad+b^2-cd+bc+d^2$. We also see that

$$\Delta_{12}+\Delta_{34}=-\Delta_{13}+\Delta_{24}=\Delta_{14}+\Delta_{23}=a^2+b^2+c^2+d^2=k.$$

We observe that this calculations give a proof the fact that the Diophantine equation (1) has always non-trivial solutions for all odd k.

PROPOSITION 3.1. *Every triangle given by (11) can be extended to a regular tetrahedron in* $\mathbb{Z}^4$.

PROOF. We may take $R((m-n)d, x, y, z)$ such that

$$ax + by + cz = (m+n)d^2 \quad \text{and} \quad x^2 + y^2 + z^2 = d^2(3m^2 - 2mn + 3n^2).$$

We can make the substitution, $x = u + a\frac{m+n}{3}$, $y = v + b\frac{m+n}{3}$ and $z = w + c\frac{m+n}{3}$ where $au + bv + cw = 0$. This substitution ensures that the first equation is satisfied. The second equation is equivalent to

$$u^2 + v^2 + w^2 = \frac{8d^2(m^2 - mn + n^2)}{3}.$$

We need the following fact

We take $(u, v, w) = \overrightarrow{OQ} = \frac{2m-4n}{3}\overrightarrow{\zeta} + \frac{2m+2n}{3}\overrightarrow{\eta}$, then by the above theorem we have

$$u^2 + v^2 + w^2 = 2d^2\left[\left(\frac{2m-4n}{3}\right)^2 + \left(\frac{2m-4n}{3}\right)\left(\frac{2m+2n}{3}\right) + \left(\frac{2m+2n}{3}\right)^2\right] \Rightarrow$$

$$u^2 + v^2 + w^2 = \frac{8d^2(m^2 - mn + n^2)}{3} \text{ and } au + bv + cw = 0. \tag{17}$$

So, we need to look when $x = u + a\frac{m+n}{3}$, $y = v + b\frac{m+n}{3}$ and $z = w + c\frac{m+n}{3}$ are integers if (u, v, w) are as above. Because of symmetry, it is enough to look at the last component; z is equal to

$$z = \frac{(2m-4n)r}{3} + \frac{(2m+2n)}{3}\frac{(r+s)}{2} + c\frac{m+n}{3} = m\frac{3r+s+c}{3} - n\frac{3r-s-c}{3},$$

or

$$z = (m-n)r + (m+n)\frac{s+c}{3}.$$

Since $2q \equiv s^2 \ (mod\ 3)$ and $q \equiv -c \ (mod\ 3)$ implies $c \equiv \pm s \ (mod\ 3)$. We may choose the vector $\overrightarrow{\zeta}$ (changing the sign if necessary) so that $s + c \equiv 0 \ (mod\ 3)$. In fact we can accomplish, by the change of sign, that two of the coordinates be integers. The third coordinate must be also an integer since $x^2 + y^2 + z^2 = d^2(3m^2 - 2mn + 3n^2)$. ■

PROPOSITION 3.2. *Suppose* $A = (x_1, x_2, x_3, x_4) \in \mathbb{Z}^4$, *different of the origin, is a point such that the number* $D = x_1^2 + x_2^2 + x_3^2 + x_4^2$ *is even. Then there exists* $B \in \mathbb{Z}^4$ *such that* OAB *is equilateral. Moreover, there exists* $C \in \mathbb{Z}^4$ *such that* $OABC$ *is a regular tetrahedron.*

PROOF. It is clear that the fact that D even translates into the fact that the number of odd numbers in the list $[x_1, x_2, x_3, x_4]$ is even. This means, we can match them in two pairs such that

the numbers in each pair are of the same parity. So, without loss of generality we may assume that x_1 and x_2 have the same parity and the same is true for x_3 and x_4. Hence the systems

$$\begin{cases} a+b=x_1 \\ \\ a-b=x_2 \end{cases}, \quad \begin{cases} c+d=x_3 \\ \\ c-d=x_4 \end{cases}$$

have solutions in integers for a, b, c, and d. Then the statement follows from the parametrization given in (10). ■

Let us observe that the two parameterizations (10) and (11) cover different sets. If we take $d=11$ and $a=5$, $b=7$ and $c=17$ we get with the above construction the regular tetrahedron $\{O,(11,5,7,17),(20,-8,4,2),(15,3,-13,9)\}$ and from (11) and Proposition 3.1 we get an essentially different regular tetrahedron $\{0,(11,5,7,17),(22,0,0,0),(11,1,19,-1)\}$.

Theorem 3.3. *Given k odd, and two different representations (1), i.e.*

$$3k^2=a^2+b^2+c^2=a'^2+b'^2+c'^2, \ \textit{with} \ \gcd(a,b,c,a',b',c')=1, \ c'>c,$$

then if we set $\Delta_{12}=\frac{a'-a}{2}$, $\Delta_{34}=\frac{a+a'}{2}$, $\Delta_{13}=\frac{b-b'}{2}$, $\Delta_{24}=\frac{b+b'}{2}$, $\Delta_{14}=\frac{c+c'}{2}$, and $\Delta_{23}=\frac{c'-c}{2}$ the equations (15) and (14) are satisfied. Moreover, the two dimensional space $\mathcal{S}$ of all vectors in $(u,v,w,t)\in\mathbb{Z}^4$, such that

$$\begin{cases} (0)u+\Delta_{34}v+\Delta_{24}w+\Delta_{23}t=0 \\ \Delta_{23}u+\Delta_{13}v+\Delta_{12}w+(0)t=0? \end{cases} \tag{18}$$

contains a family of equilateral triangles as in Proposition 2.1. Conversely, every equilateral triangle in $\mathbb{Z}^4$ can be obtained in this way up to a permutation and change of signs of the variables.

Proof. Clearly by construction we have (15). Since a, a', ... are all odd, then all numbers defined above are integers. We notice that

$$\Delta_{12}\Delta_{34}-\Delta_{13}\Delta_{24}+\Delta_{14}\Delta_{23}=(1/4)[a'^2-a^2-(b^2-b'^2)+(c'^2-c^2)]=0.$$

Let us introduce $\Delta_1:=\gcd(\Delta_{34},\Delta_{24},\Delta_{23})$ and $\Delta_2:=\gcd(\Delta_{23},\Delta_{13},\Delta_{12})$. Clearly $\gcd(\Delta_1,\Delta_2)=1$. The equations of the hyper-planes in (18) do not depend on Δ_1 and Δ_2. We see that the assumption on $\Delta_{23}>0$ insures that the equations (18) define a two dimensional space in $\mathbb{R}^4$. Moreover it is true that the map $\{(0,\Delta_{34},\Delta_{24},\Delta_{23}),(\Delta_{23},\Delta_{13},\Delta_{12},0)\}\to\mathcal{S}$ is one-to-one as long as $\Delta_{23}>0$ and $\gcd(\Delta_{12},\Delta_{34},\Delta_{13},\Delta_{24},\Delta_{23})=1$. In order to prove our claim, it suffices to show that an equilateral triangle with rational coordinates exists in $\mathcal{S}$.

We can solve the equations (18) for rational values of t and u:

$$t=-\frac{\Delta_{34}v+\Delta_{24}w}{\Delta_{23}},\ u=-\frac{\Delta_{13}v+\Delta_{12}w}{\Delta_{23}},\ \text{with } v,w\in\mathbb{Z}.$$

For infinitely many values of v and w we get integer values for t and u, but for calculation purposes even for rational values we get the quadratic form

$$QF(u,v)=\left(\frac{\Delta_{34}v+\Delta_{24}w}{\Delta_{23}}\right)^2+v^2+w^2+\left(\frac{\Delta_{13}v+\Delta_{12}w}{\Delta_{23}}\right)^2,$$

or

$$QF(u,v)=\frac{(\Delta_{34}^2+\Delta_{13}^2+\Delta_{23}^2)v^2+2(\Delta_{34}\Delta_{24}+\Delta_{13}\Delta_{12})vw+(\Delta_{24}^2+\Delta_{12}^2+\Delta_{23}^2)w^2}{\Delta_{23}^2}.$$

Using Lagrange's equality the determinant of this form (excluding the denominator) is equal to

$$-\Delta/4=(\Delta_{34}^2+\Delta_{13}^2+\Delta_{23}^2)(\Delta_{24}^2+\Delta_{12}^2+\Delta_{23}^2)-(\Delta_{34}\Delta_{24}+\Delta_{13}\Delta_{12})^2=$$

$$(\Delta_{12}\Delta_{34}-\Delta_{13}\Delta_{24}+\Delta_{23}^2)^2+(\Delta_{12}\Delta_{23}-\Delta_{34}\Delta_{23})^2+(\Delta_{13}\Delta_{23}+\Delta_{24}\Delta_{23})^2=$$

$$\Delta_{23}^2(3k^2)\Rightarrow\Delta=-3(2k\Delta_{23})^2.$$

This implies that for $v_0=-(\Delta_{34}\Delta_{24}+\Delta_{13}\Delta_{12})$ and $w_0=(\Delta_{34}^2+\Delta_{13}^2+\Delta_{23}^2)$ we get

$$QF(v_0,w_0)=3k^2(\Delta_{34}^2+\Delta_{13}^2+\Delta_{23}^2),$$

and

$$QF(v,w)=\frac{(w_0v-v_0w)^2+3k^2w^2\Delta_{23}^2}{\Delta_{23}^2w_0}.\tag{19}$$

If we denote the generic point P in the plane (18) in terms of v and w, i.e.

$$P(v,w)=(-\frac{\Delta_{13}v+\Delta_{12}w}{\Delta_{23}},v,w,-\frac{\Delta_{34}v+\Delta_{24}w}{\Delta_{23}}),$$

we observe that for two pairs of the parameters, (v,w) and (v',w'), we obtain $\Delta_{23}'=wv'-vw'$,

$$\Delta_{12}'=v(-\frac{\Delta_{13}v'+\Delta_{12}w'}{\Delta_{23}})+\frac{\Delta_{13}v+\Delta_{12}w}{\Delta_{23}}v'=\Delta_{12}\Delta_{23}'/\Delta_{23},$$

and similarly $\Delta_{13}'=\Delta_{13}\Delta_{23}'/\Delta_{23}$, etc. This implies from (13) that if $|P(v,w)|^2=|P(v',w')|^2=2k\ell$ then

$$(<P(v,w),P(v',w')>)^2=4k^2\ell^2-3k^2\left(\frac{\Delta_{23}'}{\Delta_{23}}\right)^2.$$

We see that it is enough to have $\Delta_{23}'=\ell\Delta_{23}$ to obtain an equilateral triangle. So, by (19) we need to have solutions (v,w) and (v',w') of

$$(w_0v - v_0w)^2 + 3k^2w^2\Delta_{23}^2 = (w_0v' - v_0w')^2 + 3k^2w'^2\Delta_{23}^2 = 2k\Delta_{23}w_0(wv' - vw'). \tag{20}$$

In general, the quadratic equation $x^2 + 3y^2 = n$, if it has a solution, then it has other solutions, of course other than the the trivial ones by changing the signs of x and y. For instance, we have

$$\left(\frac{x-3y}{2}\right)^2 + 3\left(\frac{x+y}{2}\right)^2 = x^2 + 3y^2 = n.$$

Let us assume that (v', w') gives this exactly this other solution:

$$\begin{cases} w_0v' - v_0w' = \frac{w_0v - v_0w + 3kw\Delta_{23}}{2} \\ \\ k\Delta_{23}w' = -\frac{w_0v - v_0w - kw\Delta_{23}}{2}. \end{cases}$$

This allows us to calculate

$$wv' - vw' = (1/2)\frac{(w_0v - v_0w)^2 + 3k^2w^2\Delta_{23}^2}{k\Delta_{23}w_0}.$$

This means that we just need to have a solution of

$$(w_0v - v_0w)^2 + 3k^2w^2\Delta_{23}^2 = 2k\Delta_{23}^2w_0\ell, \ \ell \in \mathbb{N}. \tag{21}$$

We can use the first value of ℓ which makes the prime factorization of $2kw_0\ell$ satisfy the necessary and sufficient condition for having a representation as $x^2 + 3y^2$. This condition is that all prime numbers of the form $6k - 1$ must have even exponents. Once we have such a representation we easily solve for w and then for v since $w_0 = (\Delta_{34}^2 + \Delta_{13}^2 + \Delta_{23}^2) > 0$. This proves that we always have a rational equilateral triangle in $\mathcal{S}$. By scaling it with a integer factor to eliminate all the denominators we obtain what we want.

For the last statement, we have shown the necessity of this construction at the beginning of this section. Let us observe that by permuting the variables and changing the signs we may assume that $\Delta_{23} > 0$. If $\gamma = \gcd(a, b, c, a', b', c') > 1$ we can divide all these numbers by γ. ■

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Current address: Department of Mathematics, Columbus State University, 4225 University Avenue, Columbus, GA 31907

E-mail address: math@ejionascu.edu